

## EULER–MAHONIAN STATISTICS VIA POLYHEDRAL GEOMETRY

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ABSTRACT. A variety of descent and major-index statistics have been defined for symmetric groups, hyperoctahedral groups, and their generalizations. Typically associated to pairs of such statistics is an *Euler–Mahonian distribution*, a bivariate generating function identity encoding these statistics. We use techniques from polyhedral geometry to establish new multivariate generalizations for many of the known Euler–Mahonian distributions. The original bivariate distributions are then straightforward specializations of these multivariate identities. A consequence of these new techniques are bijective proofs of the equivalence of the bivariate distributions for various pairs of statistics.

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## 1. INTRODUCTION

The symmetric group  $S_n$  is the group of permutations of  $[n] := \{1, 2, \dots, n\}$ , also realized as the Coxeter group  $A_{n-1}$  yielding symmetries of a simplex. For a permutation  $\pi \in S_n$ , the descent set is a classical object of study in combinatorics.

**Definition 1.1.** Let  $\pi \in S_n$ . The *descent set* of  $\pi$  is

$$\text{Des}(\pi) := \{j \in [n-1] : \pi(j) > \pi(j+1)\}.$$

The *descent statistic* is  $\text{des}(\pi) := \#\text{Des}(\pi)$ .

The descent statistic is encoded in the *Eulerian polynomial*  $\sum_{\pi \in S_n} t^{\text{des}(\pi)}$  and the most basic identity for Eulerian polynomials is

$$(1) \quad \sum_{k \geq 0} (k+1)^n t^k = \frac{\sum_{\pi \in S_n} t^{\text{des}(\pi)}}{(1-t)^{n+1}}.$$

Euler used this identity to *define* Eulerian polynomials in [13] which he needed in his study of what is now called the Riemann  $\zeta$ -function; one can safely assume that he was aware of the connection of his polynomials to descent statistics (see [15] which also gives connections between Eulerian polynomials and algebraic geometry).

Equation (1) has inspired a host of generalizations and extensions. The first such extension is the following  $q$ -analogue of (1), often attributed to Carlitz [10] but actually going back to MacMahon [19, Volume 2, Chapter IV, §462]. This extension involves a joint distribution of the descent statistic and the major index, defined as follows.

**Definition 1.2.** For  $\pi \in S_n$ , the *major index* of  $\pi$  is

$$\text{maj}(\pi) := \sum_{j \in \text{Des}(\pi)} j.$$

MacMahon's extension of (1) is the following, where  $[m]_q := 1 + q + q^2 + \dots + q^{m-1}$ .

**Theorem 1.3** (MacMahon).

$$\sum_{k \geq 0} [k+1]_q^n t^k = \frac{\sum_{\pi \in S_n} t^{\text{des}(\pi)} q^{\text{maj}(\pi)}}{\prod_{j=0}^n (1 - tq^j)}.$$

Note that (1) follows from Theorem 1.3 by setting  $q = 1$ . The search for further generalizations of this identity, called the *Euler–Mahonian distribution* or the *Carlitz identity*, has focused on finding new identities of the form

$$(2) \quad \sum_{k \geq 0} [k+1]_q^n t^k = \frac{\sum_{g \in G_n} t^{\text{stat}_1(g)} q^{\text{stat}_2(g)}}{\prod_{j=0}^n h_j(t, q)}$$

for various families of groups  $G_n$  and statistics  $\text{stat}_1$  and  $\text{stat}_2$  defined on elements of  $G_n$ , together with naturally occurring families of functions  $h_j(t, q)$ . This search has been successful, also producing analogous generalizations of the identities (6) and (7) discussed in the next section. To our knowledge, there are three general approaches to proving such identities:

- via combinatorial/bijective proofs in the theory of partitions and their extensions;
- via connections between permutation statistics and the theory of Coxeter groups, including connections to invariant theory and the coinvariant algebra of a Weyl group; and
- via the theory of symmetric/quasisymmetric functions.

For more information regarding the first two approaches, see the citations listed throughout this paper. For examples of the symmetric/quasisymmetric function approach, see [17, 20, 23].

Our goal is to provide new multivariate generalizations of these identities using polyhedral geometry and lattice-point enumeration; as a consequence, we obtain new proofs of two-variable identities in the form of (2). One of the benefits of the geometric approach is that it is relatively simple, the key ingredients being the triangulation of the unit cube by the braid arrangement together with careful choices of ray generators for unimodular cones. Another benefit is that bijective proofs of the equidistribution of various pairs of statistics are obtained as immediate corollaries.

As we discuss in Remark 4.3, our multivariate identities can be viewed as Hilbert series identities for various finely-graded algebras, i.e., algebras equipped with an  $\mathbb{N}^n$ -grading. A Hilbert series approach to multi-variate extensions of these identities has previously been used in [2] and subsequent papers, emphasizing the use of descent bases for coinvariant algebras. Our algebras and specializations are in some sense more straightforward than the previously considered ones, because the bivariate identities arise as simple specializations of our multivariate identities, requiring minimal or no additional substitutions and algebraic manipulations. The geometric perspective also allows us to avoid the use of straightening laws and other algebraic techniques regarding coinvariant algebras.

Our paper is structured as follows. In Section 2, we discuss analogues of (1) for generalizations of permutation groups. In Section 3, we discuss the results we will need from integer-point enumeration and polyhedral geometry. In Section 4, we use polyhedral geometry to prove Theorem 4.1; this proof serves as a model for all the proofs in the paper. We also briefly discuss connections between our approach, the theory of  $P$ -partitions, and the theory of affine semigroup algebras.

Sections 5, 6, and 7 contain our new results. In Section 5, we prove Theorems 5.7, 5.9, 5.14, and 5.15. These four theorems give new multivariate identities for negative and flag statistics in type  $B$ . In Section 6, we prove Theorems 6.3 and 6.4, new type- $D$  generating function identities. Finally, Section 7 contains theorems extending our type- $B$  results to wreath products of cyclic groups with symmetric groups.

## 2. GENERALIZED PERMUTATION GROUPS AND DESCENTS

We discuss in this section analogues of (1) for hyperoctahedral groups, type- $D$  permutation groups, and wreath products of cyclic groups with symmetric groups.

**2.1. Hyperoctahedral groups.** The hyperoctahedral group  $B_n$ , the Coxeter group yielding symmetries of a cube, consists of *signed permutations* [22], i.e., pairs  $(\pi, \epsilon)$  where  $\pi \in S_n$  and  $\epsilon \in \{\pm 1\}^n$ . For elements of  $B_n$ , there are several definitions of descents in the literature; we provide three of them here.

**Definition 2.1.** For an element  $(\pi, \epsilon) \in B_n$ , the *naturally ordered descent set* is

$$(3) \quad \text{NatDes}(\pi, \epsilon) := \{j \in \{0, 1, \dots, n-1\} : \epsilon_j \pi(j) > \epsilon_{j+1} \pi(j+1)\},$$

with the convention  $\epsilon_0 \pi(0) = 0$ . The *natural descent statistic* for  $B_n$  is  $\text{natdes}(\pi, \epsilon) := \#\text{NatDes}(\pi, \epsilon)$ .

The reason for calling this the *naturally ordered descent set* is that it uses the natural order  $-n < -n+1 < \dots < -1 < 1 < 2 < \dots < n$  on the integers. Thus, the permutation (written as a word)

$$-3 \ 2 \ -1 \ -4$$

in  $B_4$  has descents in the second and third positions. The next two definitions of descent set for  $B_n$  are special cases of descents in the context of *colored*, or *indexed*, permutations arising from wreath products.

**2.2. Wreath products.** The wreath product  $\mathbb{Z}_r \wr S_n$  of a cyclic group of order  $r$  with  $S_n$  consists of pairs  $(\pi, \epsilon)$  where  $\pi \in S_n$  and  $\epsilon \in \{\omega^0, \omega^1, \dots, \omega^{r-1}\}^n$  for  $\omega := e^{2\pi i/r}$ . Thus,  $\epsilon$  is a sequence of powers of an  $r^{\text{th}}$  root of unity; the group  $B_n$  arises as the wreath product  $\mathbb{Z}_2 \wr S_n$ . By convention, for elements of  $\mathbb{Z}_r \wr S_n$  we define additional values of  $\pi$  and  $\epsilon$  as follows:  $\pi_{n+1} := n+1$ ,  $\epsilon_{n+1} := 1$ ,  $\pi_0 := 0$ , and  $\epsilon_0 := 1$ .

We will find it convenient to use *window notation* for elements of wreath products: If  $\epsilon_j = \omega^{c_j}$ , then we will denote  $(\pi, \epsilon)$  as the *window*  $[\pi(1)^{c_1} \pi(2)^{c_2} \dots \pi(n)^{c_n}]$ . We use the notation  $j^{c_j}$  and  $(\omega^{c_j}, j)$  interchangeably for elements of  $\{\omega^0, \omega^1, \dots, \omega^{r-1}\} \times [n]$ . It is sometimes convenient to refer to  $\pi(j)^{c_j}$  as  $\pi(j)$  with *color*  $c_j$ . In [27], Steingrímsson defined the following descent set for elements of  $\mathbb{Z}_r \wr S_n$ .

**Definition 2.2.** Totally order the elements of  $\{\omega^0, \omega^1, \dots, \omega^{r-1}\} \times [n]$  by  $j^{c_j} < k^{c_k}$  if  $c_j < c_k$  or if both  $c_j = c_k$  and  $j < k$  hold. For an element  $(\pi, \epsilon) = [\pi(1)^{c_1} \pi(2)^{c_2} \dots \pi(n)^{c_n}]$  in  $\mathbb{Z}_r \wr S_n$ , *Steingrímsson's descent set* is

$$(4) \quad \text{StDes}(\pi, \epsilon) := \{j \in \{1, \dots, n\} : \pi(j)^{c_j} > \pi(j+1)^{c_{j+1}}\}.$$

*Steingrímsson's descent statistic* is  $\text{stdes}(\pi, \epsilon) := \#\text{StDes}(\pi, \epsilon)$ .

As an example, observe that with Steingrímsson's ordering we have  $\{\omega^0, \omega^1, \omega^2\} \times [3]$  ordered as

$$1^0 < 2^0 < 3^0 < 1^1 < 2^1 < 3^1 < 1^2 < 2^2 < 3^2,$$

and the permutation

$$[2^2 \ 3^2 \ 1^1]$$

has descents in positions 2 and 3.

Finally, we define the following closely-related descent set. This definition differs from Steingrímsson's both in the role played by the order of the roots of unity and in the indices where descents may occur.

**Definition 2.3.** Totally order the elements of  $\{\omega^{r-1}, \omega^{r-2}, \dots, \omega^0\} \times [n]$  by  $j^{c_j} < k^{c_k}$  if  $c_j > c_k$  or if both  $c_j = c_k$  and  $j < k$  hold. For an element  $(\pi, \epsilon) = [\pi(1)^{c_1} \pi(2)^{c_2} \dots \pi(n)^{c_n}]$  in  $\mathbb{Z}_r \wr S_n$ , the *descent set* is

$$(5) \quad \text{Des}(\pi, \epsilon) := \{j \in \{0, \dots, n-1\} : \pi(j)^{c_j} > \pi(j+1)^{c_{j+1}}\}.$$

The *descent statistic* is  $\text{des}(\pi, \epsilon) := \#\text{Des}(\pi, \epsilon)$ .

As an example, observe that with this order we have  $\{\omega^0, \omega^1, \omega^2\} \times [3]$  ordered as

$$1^2 < 2^2 < 3^2 < 1^1 < 2^1 < 3^1 < 1^0 < 2^0 < 3^0,$$

and the permutation

$$[3^2 \ 2^0 \ 1^1]$$

has descents in positions 0 and 2.

The Eulerian polynomials for wreath products are given by  $\sum_{(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n} t^{\text{des}(\pi, \epsilon)}$  where one may use either of the two wreath product descent definitions or, in the case  $r = 2$ , the natural descent statistic. The resulting analogue of (1) is

$$(6) \quad \sum_{k \geq 0} (rk + 1)^n t^k = \frac{\sum_{(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n} t^{\text{des}(\pi, \epsilon)}}{(1-t)^{n+1}}.$$

This identity appears to have been found by various authors for different descent statistics; for more details, see [9, 27].

**2.3. Type  $D$ .** One may also consider type- $D$  Eulerian polynomials stemming from the signed permutations in  $B_n$  with an even number of  $-1$ 's. Let

$$D_n := \{(\pi, \epsilon) \in B_n : \epsilon_1 \cdots \epsilon_n = 1\}.$$

The definition of  $\text{DNatDes}(\pi, \epsilon)$  and  $\text{dnatdes}(\pi, \epsilon)$  in type  $D$  is analogous to (3), except that we now use the convention  $\epsilon_0\pi(0) := -\epsilon_2\pi(2)$ . Brenti [9, Theorem 4.10] proved that

$$(7) \quad \sum_{k \geq 0} ((2k+1)^n - 2^{n-1}(B_n(k+1) - B_n(0))) t^k = \frac{\sum_{(\pi, \epsilon) \in D_n} t^{\text{dnatdes}(\pi, \epsilon)}}{(1-t)^{n+1}},$$

where  $B_n(x)$  is the  $n$ 'th Bernoulli polynomial.

### 3. A GEOMETRIC PERSPECTIVE

The forms of equations (1), (6), and (7) suggest that one should look at them geometrically as stemming from lattice-point enumeration of the cube  $[0, r]^n$  as it is partitioned in various ways; for example, (1) suggests we consider  $[0, 1]^n$  partitioned by the braid arrangement consisting of the hyperplanes  $x_j = x_k$  for  $1 \leq j < k \leq n$ . We will encounter certain simplices below, which generally look like (after a change of variables)

$$\Delta_I := \left\{ \mathbf{x} \in \mathbb{R}^n : \begin{array}{l} 0 \leq x_n \leq x_{n-1} \leq \cdots \leq x_1 \leq 1, \\ x_{j+1} < x_j \text{ if } j \in I \end{array} \right\},$$

where  $I \subseteq [n]$  is some index set, and we use the convention  $x_n > 0$  if  $n \in I$ . The vertices of  $\Delta_I$  are  $0$ ,  $\mathbf{e}_1 + \cdots + \mathbf{e}_n$ ,  $\mathbf{e}_1 + \cdots + \mathbf{e}_{n-1}$ ,  $\dots$ ,  $\mathbf{e}_1 + \mathbf{e}_2$ ,  $\mathbf{e}_1$ , where  $\mathbf{e}_j$  is the  $j$ 'th unit vector in  $\mathbb{R}^n$ . Note that  $\Delta_I$  is *unimodular*, i.e., the  $n$  edge directions at any vertex of  $\Delta_I$  generate  $\mathbb{Z}^n$ .

The *cone over*  $\Delta_I$  is the nonnegative span of  $\{(1, \mathbf{x}) \in \mathbb{R}^{n+1} : \mathbf{x} \in \Delta_I\}$ , where we encode the “new” dimension by the variable  $x_0$ , i.e.,

$$\text{cone}(\Delta_I) := \mathbb{R}_{\geq 0} \mathbf{e}_0 + \sum_{j \in I} \mathbb{R}_{> 0} (\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_j) + \sum_{j \notin I} \mathbb{R}_{\geq 0} (\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_j),$$

where the complement of  $I$  is taken in  $[n]$ . Let

$$\sigma_C(z_0, z_1, \dots, z_n) := \sum_{\mathbf{m} \in C \cap \mathbb{Z}^{n+1}} \mathbf{z}^{\mathbf{m}}$$

be the multivariate (“full”) generating function encoding the integer lattice points in a cone  $C$ , where we have used the shorthand  $\mathbf{z}^{\mathbf{m}} := z_0^{m_0} z_1^{m_1} \cdots z_n^{m_n}$ . A standard geometric-series argument (see, e.g., [5, Chapter 3]), together with the unimodularity of  $\text{cone}(\Delta_I)$ , gives the following.

**Lemma 3.1.** *Let  $\Delta_I$  be as above. Then*

$$\sigma_{\text{cone}(\Delta_I)}(z_0, z_1, \dots, z_n) = \frac{\prod_{j \in I} z_0 z_1 z_2 \cdots z_j}{\prod_{j=0}^n (1 - z_0 z_1 z_2 \cdots z_j)}.$$

We will not always use the above natural way to write the generating function of a unimodular cone, in which case we will apply the following more general lemma (for details, again see, e.g., [5, Chapter 3] or [26, Section 4.6]).

**Lemma 3.2.** *Let  $C = \sum_{j=0}^k \mathbb{R}_{\geq 0} \mathbf{v}_j + \sum_{j=k+1}^n \mathbb{R}_{> 0} \mathbf{v}_j$  be a half-open simplicial cone in  $\mathbb{R}^{n+1}$  with linearly independent generators  $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{Z}^{n+1}$ . Then*

$$\sigma_C(z_0, z_1, \dots, z_n) = \frac{\sigma_{\Pi}(z_0, z_1, \dots, z_n)}{\prod_{j=0}^n (1 - \mathbf{z}^{\mathbf{v}_j})}$$

where  $\Pi := \sum_{j=0}^k [0, 1] \mathbf{v}_j + \sum_{j=k+1}^n (0, 1] \mathbf{v}_j$ . Furthermore, the number of integer points in  $\Pi$  (and thus the number of monomials in  $\sigma_\Pi(z_0, z_1, \dots, z_n)$ ) is given by the determinant of the matrix with column vectors  $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n$ .

We refer to  $\Pi$  as the *fundamental parallelepiped* of  $C$ ; note that it depends on the choice of generators of  $C$ .

#### 4. TYPE A

We begin with a multivariate identity that specializes to Theorem 1.3. The proof of this identity, though simple, demonstrates the approach used in this paper.

**Theorem 4.1.**

$$\sum_{k \geq 0} \prod_{j=1}^n [k+1]_{z_j} z_0^k = \sum_{\pi \in S_n} \frac{\prod_{j \in \text{Des}(\pi)} z_0 z_{\pi(1)} z_{\pi(2)} \cdots z_{\pi(j)}}{\prod_{j=0}^n (1 - z_0 z_{\pi(1)} z_{\pi(2)} \cdots z_{\pi(j)})}.$$

*Proof.* Triangulate the  $n$ -cube  $[0, 1]^n$  into the disjoint union of simplices

$$\Delta_\pi := \left\{ \mathbf{x} \in \mathbb{R}^n : \begin{array}{l} 0 \leq x_{\pi(n)} \leq x_{\pi(n-1)} \leq \cdots \leq x_{\pi(1)} \leq 1, \\ x_{\pi(j+1)} < x_{\pi(j)} \text{ if } j \in \text{Des}(\pi) \end{array} \right\}$$

(one for each  $\pi \in S_n$ ). The strict inequalities determined by the descent set of  $\pi$  ensures that this triangulation is *disjoint*. By Lemma 3.1,

$$\sigma_{\text{cone}(\Delta_\pi)}(z_0, z_1, \dots, z_n) = \frac{\prod_{j \in \text{Des}(\pi)} z_0 z_{\pi(1)} z_{\pi(2)} \cdots z_{\pi(j)}}{\prod_{j=0}^n (1 - z_0 z_{\pi(1)} z_{\pi(2)} \cdots z_{\pi(j)})}.$$

On the other hand,

$$\sigma_{\text{cone}([0,1]^n)}(z_0, z_1, \dots, z_n) = \sum_{k \geq 0} \prod_{j=1}^n \left( 1 + z_j + z_j^2 + \cdots + z_j^k \right) z_0^k,$$

and the disjoint triangulation gives

$$\sigma_{\text{cone}([0,1]^n)}(z_0, z_1, \dots, z_n) = \sum_{\pi \in S_n} \sigma_{\text{cone}(\Delta_\pi)}(z_0, z_1, \dots, z_n). \quad \square$$

*Proof of Theorem 1.3.* Setting  $t := z_0$  and  $q := z_1 = z_2 = \cdots = z_n$  in Theorem 4.1 gives

$$\sum_{k \geq 0} [k+1]_q^n t^k = \frac{\sum_{\pi \in S_n} \prod_{j \in \text{Des}(\pi)} t q^j}{\prod_{j=0}^n (1 - t q^j)} = \frac{\sum_{\pi \in S_n} t^{\text{des}(\pi)} q^{\text{maj}(\pi)}}{\prod_{j=0}^n (1 - t q^j)}. \quad \square$$

*Remark 4.2.* Our approach is related to the theory of  $P$ -partitions [24, 26], which can be interpreted geometrically through triangulations of the unit cube. Given a finite poset  $P$ , the set of  $P$ -partitions can be interpreted as the integer points inside a cone admitting a nice unimodular triangulation, where the unimodular subcones are indexed by linear extensions of  $P$ . Our approach is based almost entirely on the triangulation of  $[0, 1]^n$  induced by the braid arrangement; the relationship with  $P$ -partitions is that  $[0, 1]^n$  is a truncation of the  $P$ -partition cone in the case where  $P$  is an antichain of size  $n$ . Thus,  $[0, 1]^n$  is the order polytope of the antichain of size  $n$ . As a result, our Theorem 4.1 is similar to [24, Theorem 7.1]. However, in the rest of the paper, we repeatedly choose non-unimodular generators for unimodular cones to obtain our generating function identities. To our knowledge, this approach has not been used to study  $P$ -partitions.

*Remark 4.3.* The generating function in Theorem 4.1 is the finely-graded Hilbert series for the affine semigroup algebra with semigroup given by the set of all integer points in the cone over  $[0, 1]^n$ , as discussed in [14, 21, 25]. The study of semigroup algebras arising from polyhedral cones has been an area of intense study for combinatorial commutative algebraists over the past several decades. The most important general result regarding Hilbert series for such cones is Hochster’s theorem, which states that normal affine semigroup algebras are Cohen–Macaulay [16]. The Cohen–Macaulay property forces serious constraints on single-variable specializations of the associated finely-graded Hilbert series for the algebra.

## 5. TYPE $B$

**5.1.  $q$ -analogues involving  $(k + 1)^n$ .** In the case  $G_n = B_n$ , there are several pairs of statistics resulting in  $q$ -ifications of (1). In [1], Adin, Brenti, and Roichman introduced a variety of relevant statistics. We first recall a descent statistic for elements of  $B_n$  that reads a word as if it were a permutation in the symmetric group.

**Definition 5.1.** Use the order  $-1 < \dots < -n < 1 < \dots < n$  on  $[-n, n] \setminus \{0\}$ . For an element  $(\pi, \epsilon) \in B_n$  we define the *type- $A$  descent set* as

$$\text{Des}_A(\pi, \epsilon) := \{i \in [n - 1] : \epsilon_i \pi_i > \epsilon_{i+1} \pi_{i+1}\}$$

and the *type- $A$  descent statistic* as

$$\text{des}_A(\pi, \epsilon) := \#\text{Des}_A(\pi, \epsilon).$$

The *type- $A$  major index* is

$$\text{major}_A(\pi, \epsilon) := \sum_{i \in \text{Des}_A(\pi, \epsilon)} i.$$

The second set of statistics we need are referred to as the *negative* statistics for type  $B$ .

**Definition 5.2.** For an element  $(\pi, \epsilon) \in B_n$ , we define the *negative set* of  $(\pi, \epsilon)$  to be

$$\text{Neg}(\pi, \epsilon) := \{i \in [n] : \epsilon_i = -1\}$$

and we define  $\text{neg}(\pi, \epsilon) := \#\text{Neg}(\pi, \epsilon)$ . For a sign vector  $\epsilon$ , we denote by  $\text{neg}(\epsilon)$  the number of  $-1$ ’s in  $\epsilon$ . We define the *negative descent multiset* as

$$\text{NDes}(\pi, \epsilon) := \text{Des}_A(\pi, \epsilon) \bigcup \{\pi(i) : \epsilon_i = -1\} = \text{Des}_A(\pi, \epsilon) \bigcup \text{Neg}((\pi, \epsilon)^{-1}).$$

The *negative descent statistic* is

$$\text{ndes}(\pi, \epsilon) := \#\text{NDes}(\pi, \epsilon)$$

and the *negative major index* is

$$\text{nmaj}(\pi, \epsilon) := \sum_{i \in \text{NDes}(\pi, \epsilon)} i.$$

Note that when summing to obtain the negative major index, one sums over a multiset, so there is possible repetition of the summands. Originally, the negative statistic was defined using the natural order, but the following theorem holds for the negative statistics defined with either order.

**Theorem 5.3** (Adin–Brenti–Roichman).

$$\sum_{k \geq 0} [k + 1]_q^n t^k = \frac{\sum_{(\pi, \epsilon) \in B_n} t^{\text{ndes}(\pi, \epsilon)} q^{\text{nmaj}(\pi, \epsilon)}}{(1 - t) \prod_{j=1}^n (1 - t^2 q^{2j})}.$$

The final set of statistics obtained by Adin, Brenti, and Roichman are called the *flag* statistics.

**Definition 5.4.** Use the order  $-1 < \dots < -n < 1 < \dots < n$  on  $[-n, n] \setminus \{0\}$ . For an element  $(\pi, \epsilon) \in B_n$ , define the *flag descent statistic* to be

$$\text{fdes}(\pi, \epsilon) := 2 \cdot \text{des}_A(\pi, \epsilon) + c_1,$$

where  $c_1 \in \{0, 1\}$  satisfies  $\epsilon_1 = (-1)^{c_1}$ . The *flag major index* is

$$\text{fmajor}(\pi, \epsilon) := 2 \cdot \text{major}_A(\pi, \epsilon) + \text{neg}(\pi, \epsilon).$$

The flag statistics were originally presented using the natural order, in which context the flag major index is denoted by  $\text{fmaj}$  rather than  $\text{fmajor}$ . As discussed in the remark in [1, p. 218], the following theorem holds for the flag statistics defined with either order.

**Theorem 5.5** (Adin–Brenti–Roichman).

$$\sum_{k \geq 0} [k+1]_q^n t^k = \frac{\sum_{(\pi, \epsilon) \in B_n} t^{\text{fdes}(\pi, \epsilon)} q^{\text{fmajor}(\pi, \epsilon)}}{(1-t) \prod_{j=1}^n (1-t^2 q^{2j})}.$$

**5.2. Multivariate identities.** We state and prove in this subsection multivariate generalizations of Theorems 5.3 and 5.5. Our multivariate generalizations are different than those obtained in [2] via coinvariant algebra theory.

**Definition 5.6.** Define the subset of *increasing elements* of  $B_n$ , denoted  $I_{2,n}$ , to be those elements satisfying  $\text{des}_A(\rho, \epsilon) = 0$ , i.e.,  $I_{2,n}$  contains all permutations  $(\rho, \epsilon)$  satisfying the following conditions:

- $\epsilon_j = -1$  for  $j \leq \text{neg}(\epsilon)$  and  $\epsilon_j = 1$  for  $j > \text{neg}(\epsilon)$ ;
- $\rho(1) < \rho(2) < \dots < \rho(\text{neg}(\epsilon))$ ;
- $\rho(\text{neg}(\epsilon) + 1) < \dots < \rho(n)$ .

It is straightforward that every element of  $B_n$  can be represented uniquely as  $(\rho, \epsilon)\pi = (\rho\pi, \epsilon_\pi)$  for some  $\pi \in S_n$  and  $(\rho, \epsilon) \in I_{2,n}$ , where  $\rho\pi$  is the standard composition of permutations and  $\epsilon_\pi = (\epsilon_{\pi(1)}, \epsilon_{\pi(2)}, \dots, \epsilon_{\pi(n)})$ . Thus,

$$B_n = \bigcup_{\pi \in S_n} I_{2,n}\pi.$$

Finally, observe that

$$(8) \quad \text{Neg}[(\rho, \epsilon)\pi]^{-1} = \text{Neg}((\rho, \epsilon)^{-1}) = \{\rho(j) : \epsilon_j = -1\}.$$

**Theorem 5.7.**

$$\sum_{k \geq 0} \prod_{j=1}^n [k+1]_{z_j} z_0^k = \sum_{\pi \in S_n} \sum_{(\rho, \epsilon) \in I_{2,n}} \frac{\prod_{j \in \text{Des}(\pi)} z_0 z_{\pi(1)} z_{\pi(2)} \cdots z_{\pi(j)} \prod_{j \in \text{Neg}((\rho, \epsilon)^{-1})} z_0 z_{\pi(1)} z_{\pi(2)} \cdots z_{\pi(j)}}{(1-z_0) \prod_{j=1}^n (1 - z_0^2 z_{\pi(1)}^2 \cdots z_{\pi(j)}^2)}$$

*Proof.* This proof proceeds in two stages. We first triangulate the cube  $[0, 1]^n$  into a disjoint union of simplices, then set up an indexing system for the integer points in the fundamental parallelepipeds for the cones over these simplices. Second, we bijectively associate the elements of  $B_n$  with these integer points in a way that allows us to recover, in our subsequent proofs, the descent and flag major statistics from these integer points.



We begin with the triangulation of  $\text{cone}([0, 1]^n)$  into the set of cones  $\{\text{cone}(\Delta_\pi) : \pi \in S_n\}$  found in the proof of Theorem 4.1. While  $\text{cone}(\Delta_\pi)$  is unimodular for each  $\pi$ , for this proof we use the non-unimodular ray generators

$$\mathbf{e}_0, 2(\mathbf{e}_0 + \mathbf{e}_{\pi(1)}), 2(\mathbf{e}_0 + \mathbf{e}_{\pi(1)} + \mathbf{e}_{\pi(2)}), \dots, 2(\mathbf{e}_0 + \mathbf{e}_{\pi(1)} + \dots + \mathbf{e}_{\pi(n)}).$$

There are  $2^n$  integer points in the fundamental parallelepiped for  $\text{cone}(\Delta_\pi)$  using these ray generators. Each such point can be expressed as a linear combination of the last  $n$  generators with coefficients  $\alpha_j \in \{0, \frac{1}{2}\}$ , plus a sum of shifting vectors for those integer points that need to be shifted away from the boundary of the cone. Typically one would use ray generators as shifting vectors for only those vectors that need to be shifted off the boundary. Due to the fact that  $\text{cone}(\Delta_\pi)$  is unimodular, in this case we may instead shift all the points in the fundamental parallelepiped by the minimal ray generators  $\mathbf{e}_0 + \mathbf{e}_{\pi(1)} + \dots + \mathbf{e}_{\pi(j)}$  when  $j \in \text{Des}(\pi)$ . This can also be justified by noting that the fundamental parallelepiped for  $\text{cone}(\Delta_\pi)$  with our non-unimodular ray generators is lattice-isomorphic to a half-open cube of the form  $[0, 1) \times [0, 2)^n$ .

Thus, every point  $\mathbf{p}$  in the (shifted) fundamental parallelepiped for  $\text{cone}(\Delta_\pi)$  can be uniquely expressed as

$$\mathbf{p} = \sum_{j \in \text{Des}(\pi)} \mathbf{e}_0 + \mathbf{e}_{\pi(1)} + \dots + \mathbf{e}_{\pi(j)} + \sum_{j=1}^n 2\alpha_j (\mathbf{e}_0 + \mathbf{e}_{\pi(1)} + \dots + \mathbf{e}_{\pi(j)})$$

with  $\alpha_j \in \{0, \frac{1}{2}\}$ . Associate to the point  $\mathbf{p}$  the element  $(\rho, \epsilon)\pi \in B_n$ , where  $\alpha_j = \frac{1}{2}$  if and only if  $j \in \text{Neg}((\rho, \epsilon)^{-1}) = \text{Neg}([\rho, \epsilon]\pi^{-1})$ . This correspondence creates a bijection between the elements of  $B_n$  and the (appropriately shifted) integer points in the fundamental parallelepipeds for the cones over the  $\Delta_\pi$ . Note that this bijection encodes  $I_{2,n}$  as the integer points in the fundamental parallelepiped for  $\text{cone}(\Delta_{\text{Id}})$ , where  $\text{Id}$  denotes the identity element in  $S_n$ .

Thus, we have that

$$\sigma_{\text{cone}(\Delta_\pi)}(z_0, \dots, z_n) = \sum_{(\rho, \epsilon) \in I_{2,n}} \frac{\prod_{j \in \text{Des}(\pi)} z_0 z_{\pi(1)} z_{\pi(2)} \dots z_{\pi(j)} \prod_{j \in \text{Neg}((\rho, \epsilon)^{-1})} z_0 z_{\pi(1)} z_{\pi(2)} \dots z_{\pi(j)}}{(1 - z_0) \prod_{j=1}^n (1 - z_0^2 z_{\pi(1)}^2 \dots z_{\pi(j)}^2)}.$$

This completes our proof, since from our triangulation it follows that

$$\sigma_{\text{cone}([0, 1]^n)}(z_0, \dots, z_n) = \sum_{\pi \in S_n} \sigma_{\text{cone}(\Delta_\pi)}(z_0, \dots, z_n). \quad \square$$

*Proof of Theorem 5.3.* Setting  $t = z_0$  and  $q = z_1 = \dots = z_n$  in Theorem 5.7 yields our desired form on the left-hand side of our identity, while the denominator of the right-hand side uniformly becomes

$$(1 - t) \prod_{j=1}^n (1 - t^2 q^{2j}).$$

Each element  $(\rho, \epsilon)\pi \in \bigcup_{\pi \in S_n} I_{2,n}\pi$  contributes to the numerator on the right-hand side of our identity a summand of

$$\prod_{j \in \text{Des}(\pi)} t q^j \prod_{j \in \text{Neg}([\rho, \epsilon]\pi^{-1})} t q^j.$$

Because of (8) and the fact that  $\text{Des}(\pi) = \text{Des}_A((\rho, \epsilon)\pi)$ , it follows that

$$\prod_{j \in \text{Des}(\pi)} tq^j \prod_{j \in \text{Neg}([\rho, \epsilon]\pi^{-1})} tq^j = t^{\text{ndes}(\rho(\pi, \epsilon))} q^{\text{nmaj}(\rho(\pi, \epsilon))},$$

hence our proof is complete.  $\square$

**Definition 5.8.** For an element  $\epsilon \in \{\pm 1\}^n$ , set  $\epsilon_{n+1} := 1$  and for  $j \in [n]$  set

$$a_j^\epsilon := \begin{cases} 1 & \text{if } \epsilon_j \neq \epsilon_{j+1}, \\ 0 & \text{else.} \end{cases}$$

Define  $\text{ch}(s) := \#\{j \in [n] : a_j^\epsilon = 1\}$ , the number of *sign changes* in  $\epsilon$ .

**Theorem 5.9.**

$$\sum_{k \geq 0} \prod_{j=1}^n [k+1]_{z_j} z_0^k = \sum_{(\pi, \epsilon) \in B_n} \frac{\prod_{\substack{j \in \text{Des}(\pi) \\ a_j^\epsilon = 0}} z_0^2 z_{\pi(1)}^2 z_{\pi(2)}^2 \cdots z_{\pi(j)}^2 \prod_{j: a_j^\epsilon = 1} z_0 z_{\pi(1)} z_{\pi(2)} \cdots z_{\pi(j)}}{(1-z_0) \prod_{j=1}^n \left(1 - z_0^2 z_{\pi(1)}^2 z_{\pi(2)}^2 \cdots z_{\pi(j)}^2\right)}.$$

*Proof.* As in our previous proof, this proof proceeds in two stages. The key idea in this proof, as well as in the proof of Theorem 5.5 that follows, is that the encoding of an element  $(\pi, \epsilon) \in B_n$  as an integer point is done using not the signs of  $\epsilon$ , but the sign in  $\epsilon$  given by the  $a_j^\epsilon$ 's.

We begin with the triangulation of  $\text{cone}([0, 1]^n)$  into the set of cones  $\{\text{cone}(\Delta_\pi) : \pi \in S_n\}$  found in the proof of Theorem 4.1. As in the proof of Theorem 5.7, for  $\text{cone}(\Delta_\pi)$  we use the non-unimodular ray generators

$$e_0, 2(e_0 + e_{\pi(1)}), 2(e_0 + e_{\pi(1)} + e_{\pi(2)}), \dots, 2(e_0 + e_{\pi(1)} + \cdots + e_{\pi(n)}).$$

In this case, we do not want to shift every integer point in the fundamental parallelepiped for  $\text{cone}(\Delta_\pi)$  by a uniform shifting vector; we instead only want to shift the points on the boundary of the cone, and those by the appropriate non-unimodular ray generators. Hence we represent every such point  $\mathbf{p}$  uniquely with a vector  $\alpha \in \{0, \frac{1}{2}\}^n$  of coefficients for

$$\mathbf{p} = \sum_{j=1}^n 2\alpha_j (e_0 + e_{\pi(1)} + \cdots + e_{\pi(j)}) + \sum_{\substack{j \in \text{Des}(\pi) \\ \alpha_j = 0}} 2(e_0 + e_{\pi(1)} + \cdots + e_{\pi(j)}).$$

We may then associate to the point  $\mathbf{p}$  the element  $(\pi, \epsilon) \in B_n$  where  $\pi$  is the same as the index on  $\text{cone}(\Delta_\pi)$  and  $\epsilon$  is defined by  $a_j^\epsilon = 2\alpha_j$ . This bijectively relates  $B_n$  to the (possibly shifted) integer points in the fundamental parallelepipeds of the cones over the  $\Delta_\pi$ 's. Thus, we have that

$$\sigma_{\text{cone}(\Delta_\pi)}(z_0, \dots, z_n) = \sum_{(\pi, \epsilon) \in B_n} \frac{\prod_{\substack{j \in \text{Des}(\pi) \\ a_j^\epsilon = 0}} z_0^2 z_{\pi(1)}^2 z_{\pi(2)}^2 \cdots z_{\pi(j)}^2 \prod_{j: a_j^\epsilon = 1} z_0 z_{\pi(1)} z_{\pi(2)} \cdots z_{\pi(j)}}{(1-z_0) \prod_{j=1}^n \left(1 - z_0^2 z_{\pi(1)}^2 \cdots z_{\pi(j)}^2\right)}.$$

(Note that in the sum on the right-hand side,  $\pi$  is fixed while  $\epsilon$  varies.) This completes our proof, since from our triangulation it follows that

$$\sigma_{\text{cone}([0,1]^n)}(z_0, \dots, z_n) = \sum_{\pi \in S_n} \sigma_{\text{cone}(\Delta_\pi)}(z_0, \dots, z_n). \quad \square$$

*Proof of Theorem 5.5.* Setting  $t = z_0$  and  $q = z_1 = \dots = z_n$  in Theorem 5.7 yields our desired form on the left-hand side of our identity, while the denominator of the right-hand side uniformly becomes

$$(1-t) \prod_{j=1}^n (1-t^2 q^{2j}).$$

Each element  $(\pi, \epsilon) \in B_n$  contributes to the numerator on the right-hand side of our identity a summand of

$$\prod_{\substack{j \in \text{Des}(\pi) \\ a_j^\epsilon = 0}} t^2 q^{2j} \prod_{j: a_j^\epsilon = 1} t q^j.$$

Therefore, our proof will be complete once we prove that

$$(9) \quad \text{fdes}(\pi, \epsilon) = \sum_{\substack{j \in \text{Des}(\pi) \\ a_j^\epsilon = 0}} 2 + \sum_{j: a_j^\epsilon = 1} 1$$

and

$$(10) \quad \text{fmajor}(\pi, \epsilon) = \sum_{\substack{j \in \text{Des}(\pi) \\ a_j^\epsilon = 0}} 2j + \sum_{j: a_j^\epsilon = 1} j.$$

A descent in position  $j$  of  $(\pi, \epsilon)$  can arise for one of three reasons: either  $\epsilon_j = 1$  while  $\epsilon_{j+1} = -1$ , or  $\epsilon_j = \epsilon_{j+1}$  and  $j \in \text{Des}(\pi)$ , or  $j = 1$  and  $\epsilon_1 = -1$ . When viewing  $(\pi, \epsilon)$  in window notation and reading right to left, every descent of the first kind is the result of two sign changes, one where a  $-1$  is introduced, and one where a  $1$  is reintroduced. Descents of the second kind are caused when there is no sign change but a descent in  $\pi$  is present. Descents of the third kind are caused by an “incomplete” sign change in  $\epsilon$ , where a  $-1$  is introduced when reading window notation right to left, but not returned to a  $1$ . The equality in (9) follows immediately from these observations.

To prove (10), observe that the expression  $\sum_{\substack{j \in \text{Des}(\pi) \\ a_j^\epsilon = 0}} 2j$  contributes to the type- $A$  major index for

all descents that do not involve a sign change. It is then straightforward to see that

$$\sum_{j: a_j^\epsilon = 1} j = \text{neg}(\pi, \epsilon) + \sum_{\substack{j \in \text{Des}(\pi, \epsilon) \\ \epsilon_j \neq \epsilon_{j+1}}} 2j,$$

since on the right-hand side, each descent in  $(\pi, \epsilon)$  given by a sign change contributes twice the descent position plus the number of consecutive negative signs to the right of the descent position. As all negative signs are counted once in this representation, and all descent positions are counted once in each of the two summands, the right-hand side of (10) is equal to  $2 \cdot \text{major}_A(\pi, \epsilon) + \text{neg}(\pi, \epsilon)$ . This completes our proof.  $\square$

In the original work of Adin, Brenti, and Roichman [1], it was left as an open question to give a bijective proof of the equidistribution of the pairs of statistics  $(\text{ndes}, \text{nmaj})$  and  $(\text{fdes}, \text{fmajor})$ ; a combinatorial proof leading to an implicit bijection was given by Lai and Petersen in [18]. The proofs of Theorems 5.7 and 5.9 together yield this bijective proof.

**Corollary 5.10** (Adin–Brenti–Roichman).

$$\sum_{(\pi, \epsilon) \in B_n} t^{\text{ndes}(\pi, \epsilon)} q^{\text{nmaj}(\pi, \epsilon)} = \sum_{(\pi, \epsilon) \in B_n} t^{\text{fdes}(\pi, \epsilon)} q^{\text{fmaj}(\pi, \epsilon)}$$

*Proof.* Our proof arises bijectively from the indexing of integer points in fundamental parallelepipies for  $\text{cone}(\Delta_\pi)$  found in the proofs of Theorems 5.7 and 5.9. To the element  $(\rho, \epsilon)\pi \in \bigcup_{\pi \in S_n} I_{2,n}\pi$  we associated the integer point

$$\mathbf{p} = \sum_{j \in \text{Des}(\pi)} \mathbf{e}_0 + \mathbf{e}_{\pi(1)} + \cdots + \mathbf{e}_{\pi(j)} + \sum_{j=1}^n 2\alpha_j (\mathbf{e}_0 + \mathbf{e}_{\pi(1)} + \cdots + \mathbf{e}_{\pi(j)})$$

where

$$\alpha_j = \begin{cases} 0 & \text{if } j \notin \text{Neg}((\rho, \epsilon)^{-1}), \\ \frac{1}{2} & \text{if } j \in \text{Neg}((\rho, \epsilon)^{-1}). \end{cases}$$

Representing  $\mathbf{p}$  as

$$\mathbf{p} = \sum_{j=1}^n 2\beta_j (\mathbf{e}_0 + \mathbf{e}_{\pi(1)} + \cdots + \mathbf{e}_{\pi(j)}) + \sum_{\substack{j \in \text{Des}(\pi) \\ \beta_j = 0}} 2(\mathbf{e}_0 + \mathbf{e}_{\pi(1)} + \cdots + \mathbf{e}_{\pi(j)})$$

where  $\beta_j \in \{0, \frac{1}{2}\}$ , we associated to  $\mathbf{p}$  the integer point  $(\pi, \epsilon) \in B_n$  where  $\pi$  is the same as the index on  $\text{cone}(\Delta_\pi)$  and  $\epsilon$  is defined by  $a_j^\epsilon = 2\beta_j$ . This yields an explicit bijection from  $B_n$  to itself that preserves the pairs of statistics  $(\text{ndes}, \text{nmaj})$  and  $(\text{fdes}, \text{fmaj})$ .  $\square$

**5.3.  $q$ -analogues involving  $(2k+1)^n$ .** Recall Definition 2.1 which introduced  $\text{NatDes}(\pi, \epsilon)$  and  $\text{natdes}(\pi, \epsilon)$ . Further, for an element  $(\pi, \epsilon) \in B_n$ , the *naturally ordered major index* is

$$\text{natmaj}(\pi, \epsilon) := \sum_{i \in \text{NatDes}(\pi, \epsilon)} i.$$

Chow and Gessel [11, Equation (26)] proved the following hyperoctahedral analogue of Theorem 1.3:

**Theorem 5.11** (Chow–Gessel).

$$\sum_{k \geq 0} ([k+1]_q + s[k]_q)^n t^k = \frac{\sum_{\pi \in S_n, \epsilon \in \{\pm 1\}^n} s^{\text{neg}(\epsilon)} t^{\text{natdes}(\pi, \epsilon)} q^{\text{natmaj}(\pi, \epsilon)}}{\prod_{j=0}^n (1 - tq^j)}.$$

The special case  $q = 1$  is due to Brenti [9, Theorem 3.4]. Chow and Gessel also showed in [11] how Theorem 5.11 implies other versions of “ $q$ -Eulerian polynomials” of type  $B$  involving a flag major statistic using the natural order, such as the following.

**Definition 5.12.** Use the order  $-n < \cdots < -1 < 1 < \cdots < n$  on  $[-n, n] \setminus \{0\}$ . We define the *natural type- $A$  descent set* as

$$\text{NatDes}_A(\pi, \epsilon) := \{i \in [n-1] : \epsilon_i \pi_i > \epsilon_{i+1} \pi_{i+1}\}$$

and the *natural type- $A$  major index* as

$$\text{natmajor}_A(\pi, \epsilon) := \sum_{i \in \text{NatDes}_A(\pi, \epsilon)} i.$$

The *natural flag major index* is

$$\text{natfmaj}(\pi, \epsilon) := 2 \cdot \text{natmajor}_A(\pi, \epsilon) + \text{neg}(\pi, \epsilon).$$

**Theorem 5.13** (Chow–Gessel).

$$\sum_{k \geq 0} [2k+1]_q^n t^k = \frac{\sum_{(\pi, \epsilon) \in B_n} t^{\text{natdes}(\pi, \epsilon)} q^{\text{natfmaj}(\pi, \epsilon)}}{\prod_{j=0}^n (1 - tq^{2j})}.$$

While Theorems 5.11 and 5.13 are equivalent via a change of variables, the geometric perspective illustrates how these two theorems arise as specializations of two distinct multivariate generating function identities.

**5.4. Multivariate identities.** For the type- $B$  extension of Theorem 4.1 generalizing Theorem 5.11, we introduce the variables  $z_{\pm j}$  to keep track of the positive/negative  $j$ th component of a lattice point, respectively, and the variables  $w_{\pm j}$  to indicate whether the  $j$ th coordinate of our point is nonnegative or negative.

**Theorem 5.14.**

$$\sum_{k \geq 0} \prod_{j=1}^n \left( w_j [k+1]_{z_j} + w_{-j} z_{-j}^{-1} [k]_{z_{-j}^{-1}} \right) z_0^k = \sum_{(\pi, \epsilon) \in B_n} \prod_{\epsilon_j=1} w_j \prod_{\epsilon_j=-1} z_{-j}^{-1} w_{-j} \frac{\prod_{j \in \text{NatDes}(\pi, \epsilon)} z_0 z_{\epsilon_1 \pi(1)}^{\epsilon_1} z_{\epsilon_2 \pi(2)}^{\epsilon_2} \cdots z_{\epsilon_j \pi(j)}^{\epsilon_j}}{\prod_{j=0}^n \left( 1 - z_0 z_{\epsilon_1 \pi(1)}^{\epsilon_1} z_{\epsilon_2 \pi(2)}^{\epsilon_2} \cdots z_{\epsilon_j \pi(j)}^{\epsilon_j} \right)}.$$

Note that Theorem 4.1 is the special case  $w_{-1} = \cdots = w_{-n} = 0$ ,  $w_1 = \cdots = w_n = 1$  of Theorem 5.14.

*Proof.* Triangulate the  $n$ -cube  $[-1, 1]^n$  by first disjointly decomposing  $[-1, 1]^n$  into the cubes

$$\square_\epsilon := \left\{ \mathbf{x} \in \mathbb{R}^n : \begin{array}{ll} 0 \leq \epsilon_j x_j \leq 1 & \text{if } \epsilon_j = 1, \\ 0 < \epsilon_j x_j \leq 1 & \text{if } \epsilon_j = -1 \end{array} \right\}$$

where there is one cube for every  $\epsilon \in \{\pm 1\}^n$ . Each of these cubes then gets triangulated into the disjoint simplices

$$\Delta_{\pi, \epsilon} := \left\{ \mathbf{x} \in \mathbb{R}^n : \begin{array}{l} 0 \leq \epsilon_n x_{\pi(n)} \leq \epsilon_{n-1} x_{\pi(n-1)} \leq \cdots \leq \epsilon_1 x_{\pi(1)} \leq 1 \\ \epsilon_{j+1} x_{\pi(j+1)} < \epsilon_j x_{\pi(j)} \text{ if } j \in \text{NatDes}(\pi, \epsilon) \end{array} \right\} \subseteq \square_\epsilon$$

(recall that we set  $\epsilon_0 = 0$ ). A lattice point  $\mathbf{m} \in \text{cone}(\square_\epsilon)$  gets encoded by the monomial

$$z_0^{m_0} \prod_{\epsilon_j=-1} w_{-j} z_{-j}^{m_j} \prod_{\epsilon_j=1} w_j z_j^{m_j}.$$

Note that for each  $j$ , exactly one of  $z_j$  and  $z_{-j}$  appears in this formula. Since  $\text{cone}(\Delta_{\pi, \epsilon}) \subseteq \text{cone}(\square_\epsilon)$  (and so for every  $j$ , the sign of the  $j$ th coordinate of any point in  $\text{cone}(\Delta_{\pi, \epsilon})$  is the same), we can adapt Lemma 3.1 to see that

$$\sigma_{\text{cone}(\Delta_{\pi, \epsilon})}(z_0, z_{\pm 1}, \dots, z_{\pm n}, w_{\pm 1}, \dots, w_{\pm n}) = \frac{\prod_{\epsilon_j=1} w_j \prod_{\epsilon_j=-1} z_{-j}^{-1} w_{-j} \prod_{j \in \text{NatDes}(\pi, \epsilon)} z_0 z_{\epsilon_1 \pi(1)}^{\epsilon_1} z_{\epsilon_2 \pi(2)}^{\epsilon_2} \cdots z_{\epsilon_j \pi(j)}^{\epsilon_j}}{\prod_{j=0}^n \left( 1 - z_0 z_{\epsilon_1 \pi(1)}^{\epsilon_1} z_{\epsilon_2 \pi(2)}^{\epsilon_2} \cdots z_{\epsilon_j \pi(j)}^{\epsilon_j} \right)}.$$

Again note that for each  $j$ , either  $z_j$  or  $z_{-j}$  appears in this formula, but not both. On the other hand,

$$\sigma_{\text{cone}([-1,1]^n)}(z_0, z_{\pm 1}, \dots, z_{\pm n}, w_{\pm 1}, \dots, w_{\pm n}) = \sum_{k \geq 0} \prod_{j=1}^n \left( w_{-j} z_{-j}^{-k} + w_{-j} z_{-j}^{-(k-1)} + \dots + w_{-j} z_{-j}^{-1} + w_j + w_j z_j + w_j z_j^2 + \dots + w_j z_j^k \right) z_0^k,$$

and the disjoint triangulation gives

$$\sigma_{\text{cone}([-1,1]^n)}(z_0, z_{\pm 1}, \dots, z_{\pm n}, w_{\pm 1}, \dots, w_{\pm n}) = \sum_{(\pi, \epsilon) \in B_n} \sigma_{\text{cone}(\Delta_\pi)}(z_0, z_{\pm 1}, \dots, z_{\pm n}, w_{\pm 1}, \dots, w_{\pm n}).$$

□

*Proof of Theorem 5.11.* Setting  $t := z_0$ ,  $q := z_1 = \dots = z_n = z_{-1}^{-1} = \dots = z_{-n}^{-1}$ ,  $s := w_{-1} = \dots = w_{-n}$ , and  $w_1 = \dots = w_n = 1$  in Theorem 5.14 gives

$$\begin{aligned} \sum_{k \geq 0} ([k+1]_q + s q [k]_q)^n t^k &= \frac{\sum_{(\pi, \epsilon) \in B_n} \prod_{\epsilon_j = -1} s q \prod_{j \in \text{NatDes}(\pi, \epsilon)} t q^j}{\prod_{j=0}^n (1 - t q^j)} \\ &= \frac{\sum_{(\pi, \epsilon) \in B_n} s^{\text{neg}(\epsilon)} t^{\text{natdes}(\pi, \epsilon)} q^{\text{natmaj}(\pi, \epsilon) + \text{neg}(\pi, \epsilon)}}{\prod_{j=0}^n (1 - t q^j)}. \end{aligned}$$

Replacing  $s$  by  $\frac{s}{q}$  yields Theorem 5.11. □

Our multivariate generalization of Theorem 5.13 is the following. Recall from Definition 5.8 the notation  $\text{ch}(\epsilon)$  for the number of sign changes in  $\epsilon$  and the notation  $a_j^\epsilon$  to keep track of where sign changes occur.

**Theorem 5.15.**

$$\begin{aligned} \sum_{k \geq 0} \prod_{j=1}^n [2k+1]_{z_j} z_0^k &= \sum_{(\pi, \epsilon) \in B_n} \frac{z_0^{\lceil \text{ch}(\epsilon)/2 \rceil} \prod_{j: a_j^\epsilon = 1} z_{\pi(1)} z_{\pi(2)} \dots z_{\pi(j)} \prod_{\substack{j \in \text{Des}(\pi) \\ a_j^\epsilon = 0}} z_0 z_{\pi(1)}^2 z_{\pi(2)}^2 \dots z_{\pi(j)}^2}{\prod_{j=0}^n \left( 1 - z_0 z_{\pi(1)}^2 z_{\pi(2)}^2 \dots z_{\pi(j)}^2 \right)}. \end{aligned}$$

*Proof.* We begin by triangulating  $[0, 2]^n$  into the disjoint simplices

$$\Delta_\pi := \left\{ \mathbf{x} \in \mathbb{R}^n : \begin{array}{l} 0 \leq x_{\pi(n)} \leq x_{\pi(n-1)} \leq \dots \leq x_{\pi(1)} \leq 2, \\ x_{\pi(j+1)} < x_{\pi(j)} \text{ if } j \in \text{Des}(\pi) \end{array} \right\}$$

(one for each  $\pi \in S_n$ ). As before, the strict inequalities determined by the descent set of  $\pi$  ensures that this triangulation is disjoint.

Unlike the cones produced by coning over the simplices in our triangulation of  $[0, 1]^n$ , the cones arising from this triangulation of  $[0, 2]^n$  are not unimodular. By Lemma 3.2, the integer-point transform of  $\text{cone}(\Delta_\pi)$  can be expressed as a rational function with denominator

$$\prod_{j=0}^n \left( 1 - z_0 z_{\pi(1)}^2 z_{\pi(2)}^2 \dots z_{\pi(j)}^2 \right),$$

i.e., where the displayed exponent vectors are the ray generators for this cone. As the determinant of the matrix formed by the ray generators of  $\text{cone}(\Delta_\pi)$  is  $2^n$ , there are  $2^n$  integer points in the fundamental parallelepiped of  $\text{cone}(\Delta_\pi)$ . It is a straightforward observation that there are  $2^n$  such integer points formed by taking linear combinations of the ray generators for the cone with coefficients 0 and  $\frac{1}{2}$ . We will use the following notation to denote such an integer point; for  $\alpha_j \in \{0, \frac{1}{2}\}$ ,

$$\mathbf{p} = \alpha_0 \mathbf{e}_0 + \sum_{j=1}^n \alpha_j (\mathbf{e}_0 + 2\mathbf{e}_{\pi(1)} + 2\mathbf{e}_{\pi(2)} + \cdots + 2\mathbf{e}_{\pi(j)}) .$$

To determine the numerator of  $\sigma_{\text{cone}(\Delta_\pi)}(z_0, \dots, z_n)$ , we must shift some of these interior points off of the boundary of  $\text{cone}(\Delta_\pi)$ . It is a simple exercise to show that the point  $\mathbf{p}$  must be shifted by the vector

$$\sum_{j:\alpha_j=0} \mathbf{e}_0 + 2\mathbf{e}_{\pi(1)} + 2\mathbf{e}_{\pi(2)} + \cdots + 2\mathbf{e}_{\pi(j)} .$$

Thus,

$$\begin{aligned} \sigma_{\text{cone}(\Delta_\pi)}(z_0, z_1, \dots, z_n) = & \frac{z_0^{\frac{1}{2}\#\{j \in [0, n] : \alpha_j = \frac{1}{2}\}} \prod_{j:\alpha_j=\frac{1}{2}} z_{\pi(1)} z_{\pi(2)} \cdots z_{\pi(j)} \prod_{\substack{j \in \text{Des}(\pi) \\ \alpha_j=0}} z_0^2 z_{\pi(1)}^2 z_{\pi(2)}^2 \cdots z_{\pi(j)}^2}{\prod_{j=0}^n (1 - z_0^2 z_{\pi(1)}^2 z_{\pi(2)}^2 \cdots z_{\pi(j)}^2)} . \end{aligned}$$

Note that  $\frac{1}{2}\#\{j \in [0, n] : \alpha_j = \frac{1}{2}\}$  is always an integer, as this is the coefficient of  $\mathbf{e}_0$  in the resulting integer point.

We now associate to the element  $(\pi, \epsilon) \in B_n$  the integer point

$$\alpha_0 \mathbf{e}_0 + \sum_{j=1}^n \frac{a_j^\epsilon}{2} (\mathbf{e}_0 + 2\mathbf{e}_{\pi(1)} + 2\mathbf{e}_{\pi(2)} + \cdots + 2\mathbf{e}_{\pi(j)}) + \sum_{\substack{j \in \text{Des}(\pi) \\ a_j^\epsilon=0}} (\mathbf{e}_0 + 2\mathbf{e}_{\pi(1)} + 2\mathbf{e}_{\pi(2)} + \cdots + 2\mathbf{e}_{\pi(j)})$$

in  $\text{cone}(\Delta_\pi)$ , where  $\alpha_0 = 0$  or  $\frac{1}{2}$  as needed to make  $\alpha_0 + \sum_{j=1}^n \frac{a_j^\epsilon}{2}$  an integer. Through this association,

the set of sign vectors  $\{\pm 1\}^n$  parametrizes the integer points in the fundamental parallelepiped for  $\text{cone}(\Delta_\pi)$ . This parametrization is bijective, and the coefficient of  $\mathbf{e}_0$  in the first two terms of the sum above is equal to both  $\lceil \text{ch}(\epsilon)/2 \rceil$  and  $\frac{1}{2}\#\{j \in [0, n] : \alpha_j = \frac{1}{2}\}$ . Thus, our proof is complete following the observation that

$$\sigma_{\text{cone}([0, 2]^n)}(z_0, \dots, z_n) = \sum_{\pi \in S_n} \sigma_{\text{cone}(\Delta_\pi)}(z_0, \dots, z_n) . \quad \square$$

*Proof of Theorem 5.13.* The pairs of statistics  $(\text{natdes}, \text{natfmaj})$  and  $(\text{des}, \text{fmajor})$  are equidistributed in  $B_n$ ; this is a consequence of the bijection mapping every permutation  $(\pi, \epsilon) \in B_n$  to the permutation where the  $\pi(j)$  for  $\epsilon_j = -1$  have been reversed in order. Hence, we may prove our theorem using the second pair of statistics.

Setting  $t = z_0$  and  $q = z_1 = \cdots = z_n$  in Theorem 5.15 yields our desired form on the left-hand side, while the denominator of the right-hand side uniformly becomes  $\prod_{j=0}^n (1 - tq^{2j})$ . Each

element  $(\pi, \epsilon) \in B_n$  contributes a summand of

$$t^{\lceil \text{ch}(\epsilon)/2 \rceil} \prod_{j: a_j^\epsilon = 1} q^j \prod_{\substack{j \in \text{Des}(\pi) \\ a_j^\epsilon = 0}} tq^{2j}$$

to the numerator of the right-hand side. Hence, we need to prove

$$(11) \quad \text{des}(\pi, \epsilon) = \lceil \text{ch}(\epsilon)/2 \rceil + \#\{j \in \text{Des}(\pi) : a_j^\epsilon = 0\},$$

and

$$(12) \quad \text{fmajor}(\pi, \epsilon) = \sum_{j: a_j^\epsilon = 1} j + \sum_{\substack{j \in \text{Des}(\pi) \\ a_j^\epsilon = 0}} 2j.$$

As explained in the proof of Theorem 5.5, a descent in position  $j$  of  $(\pi, \epsilon)$  can arise for one of three reasons: either  $\epsilon_j = 1$  while  $\epsilon_{j+1} = -1$ , or  $\epsilon_j = \epsilon_{j+1}$  and  $j \in \text{Des}(\pi)$ , or  $j = 1$  and  $\epsilon_1 = -1$ . When viewing  $(\pi, \epsilon)$  in window notation and reading right to left, every descent of the first kind is the result of two sign changes, one where a  $-1$  is introduced, and one where a  $1$  is reintroduced. Descents of the second kind are caused when there is no sign change but a descent in  $\pi$  is present. Descents of the third kind are caused by an “incomplete” sign change in  $\epsilon$ , where a  $-1$  is introduced when reading window notation right to left, but not returned to a  $1$ . The equality in (11) follows immediately from these observations.

To prove (12), observe that  $\sum_{\substack{j \in \text{Des}(\pi) \\ a_j^\epsilon = 0}} 2j$  contributes to the type- $A$  major index for all descents

that do not involve a sign change. It is then straightforward to see that

$$\sum_{j: a_j^\epsilon = 1} j = \text{neg}(\pi, \epsilon) + \sum_{\substack{j \in \text{Des}(\pi, \epsilon) \\ \epsilon_j \neq \epsilon_{j+1}}} 2j,$$

since on the right-hand side, each descent in  $(\pi, \epsilon)$  given by a sign change contributes twice the descent position plus the number of consecutive negative signs to the right of the descent position. As all negative signs are counted once in this representation, and all descent positions are counted once in each of the two summands, the right-hand side of (12) is equal to  $2 \cdot \text{maj}_A(\pi, \epsilon) + \text{neg}(\pi, \epsilon)$ . This completes our proof.  $\square$

## 6. TYPE $D$

**6.1.  $q$ -analogues involving  $(k+1)^n$ .** Moving on to the groups  $D_n$ , we recall the following definition of type- $D$  negative statistics [6].

**Definition 6.1.** Using the order  $-1 < \dots < -n < 1 < \dots < n$  on  $[-n, n] \setminus \{0\}$ , for an element  $(\pi, \epsilon) \in D_n$ , we define  $\text{Neg}(\pi, \epsilon)$ ,  $\text{neg}(\pi, \epsilon)$ ,  $\text{Des}_A(\pi, \epsilon)$ , and  $\text{des}_A(\pi, \epsilon)$  as for the group  $B_n$ . We define the *type- $D$  negative descent multiset* as

$$\begin{aligned} \text{DNDes}(\pi, \epsilon) &:= \text{Des}_A(\pi, \epsilon) \bigcup \{\pi(i) - 1 : \epsilon_i = -1\} \setminus \{0\} \\ &= \text{Des}_A(\pi, \epsilon) \bigcup \{j - 1 : j \in \text{Neg}((\pi, \epsilon)^{-1}) \setminus \{1\}\}. \end{aligned}$$

The *type- $D$  negative descent statistic* is

$$\text{dndes}(\pi, \epsilon) := \#\text{DNDes}(\pi, \epsilon).$$



The *type-D negative major index* is

$$\text{dnmaj}(\pi, \epsilon) := \sum_{i \in \text{DNDes}(\pi, \epsilon)} i.$$

Again in this case, Biagioli [6] defined  $\text{dnmaj}$  and  $\text{dndes}$  using the natural order, but the following theorem holds for either definition.

**Theorem 6.2** (Biagioli).

$$\sum_{k \geq 0} [k+1]_q^n t^k = \frac{\sum_{(\pi, \epsilon) \in D_n} t^{\text{dndes}(\pi, \epsilon)} q^{\text{dnmaj}(\pi, \epsilon)}}{(1-t)(1-tq^n) \prod_{i=1}^{n-1} (1-t^2 q^{2i})}.$$

**6.2. Multivariate identities.** Our multivariate generalization of Theorem 6.2 is as follows. Let  $I_{2,n}^* \subseteq I_{2,n}$  denote the elements  $(\rho, \epsilon) \in I_{2,n}$  satisfying  $\epsilon_1 \epsilon_2 \cdots \epsilon_n = 1$ . It is straightforward to show that

$$D_n = \bigcup_{\pi \in S_n} I_{2,n}^* \pi.$$

**Theorem 6.3.**

$$\sum_{k \geq 0} \prod_{j=1}^n [k+1]_{z_j} z_0^k = \sum_{\pi \in S_n} \sum_{(\rho, \epsilon) \in I_{2,n}^*} \frac{\prod_{j \in \text{Des}(\pi)} z_0 z_{\pi(1)} z_{\pi(2)} \cdots z_{\pi(j)} \prod_{j \in \text{Neg}((\rho, \epsilon)^{-1}) \setminus \{1\}} z_0 z_{\pi(1)} z_{\pi(2)} \cdots z_{\pi(j-1)}}{(1-z_0)(1-z_0 z_{\pi(1)} \cdots z_{\pi(n)}) \prod_{j=1}^{n-1} (1-z_0^2 z_{\pi(1)}^2 \cdots z_{\pi(j)}^2)}$$

*Proof.* We begin with the triangulation of  $\text{cone}([0, 1]^n)$  into the set of cones  $\{\text{cone}(\Delta_\pi) : \pi \in S_n\}$  found in the proof of Theorem 4.1. For  $\text{cone}(\Delta_\pi)$  we use the non-unimodular ray generators

$$\mathbf{e}_0, 2(\mathbf{e}_0 + \mathbf{e}_{\pi(1)}), \dots, 2(\mathbf{e}_0 + \mathbf{e}_{\pi(1)} + \cdots + \mathbf{e}_{\pi(n-1)}), \mathbf{e}_0 + \mathbf{e}_{\pi(1)} + \cdots + \mathbf{e}_{\pi(n)}.$$

There are  $2^{n-1}$  integer points in the fundamental parallelepiped for  $\text{cone}(\Delta_\pi)$  using these ray generators. Each such point can be expressed as a linear combination of the middle  $n-1$  generators with coefficients  $\alpha_j \in \{0, \frac{1}{2}\}$ , plus a sum of shifting vectors for those integer points that need to be shifted away from the boundary of the cone. As in our proof of Theorem 5.7, we will shift all the points in the fundamental parallelepiped by the minimal ray generators  $\mathbf{e}_0 + \mathbf{e}_{\pi(1)} + \cdots + \mathbf{e}_{\pi(j)}$  when  $j \in \text{Des}(\pi)$ . This can again be justified by noting that the fundamental parallelepiped for  $\text{cone}(\Delta_\pi)$  with our non-unimodular ray generators is lattice-isomorphic to a half-open cube of the form  $[0, 1) \times [0, 2)^{n-1} \times [0, 1)$ .

Thus, every point  $\mathbf{p}$  in the (shifted) fundamental parallelepiped for  $\text{cone}(\Delta_\pi)$  can be uniquely expressed as

$$\mathbf{p} = \sum_{j \in \text{Des}(\pi)} \mathbf{e}_0 + \mathbf{e}_{\pi(1)} + \cdots + \mathbf{e}_{\pi(j)} + \sum_{j=1}^{n-1} 2\alpha_j (\mathbf{e}_0 + \mathbf{e}_{\pi(1)} + \cdots + \mathbf{e}_{\pi(j)})$$

with  $\alpha_j \in \{0, \frac{1}{2}\}$ . Associate to the point  $\mathbf{p}$  the element  $(\rho, \epsilon)\pi \in D_n$ , where  $\alpha_j = \frac{1}{2}$  if and only if  $j+1 \in \text{Neg}((\rho, \epsilon)^{-1}) = \text{Neg}([\rho, \epsilon]\pi^{-1})$ . This correspondence creates a bijection between the elements of  $D_n$  and the (appropriately shifted) integer points in the fundamental parallelepipeds for the cones over the  $\Delta_\pi$ ; in this case the bijection is less obvious than in the case for  $B_n$ . Our choice of  $(\rho, \epsilon)\pi$  associated to  $\mathbf{p}$  is unique because the condition  $\alpha_j = \frac{1}{2}$  if and only if  $j+1 \in \text{Neg}((\rho, \epsilon)^{-1}) =$

$\text{Neg}([\rho, \epsilon]\pi^{-1})$  determines the signs placed on the letters  $\{2, 3, \dots, n\}$  when  $(\rho, \epsilon)\pi$  is written in window notation. Hence,  $\epsilon_1$  is determined from these  $n-1$  signs and the fact that  $\epsilon_1 \cdots \epsilon_n = 1$ . Note as before that this bijection encodes  $I_{2,n}^*$  as the integer points in the fundamental parallelepiped for  $\text{cone}(\Delta_{\text{Id}})$ . Thus

$$\sigma_{\text{cone}(\Delta_\pi)}(z_0, \dots, z_n) = \sum_{(\rho, \epsilon) \in I_{2,n}^*} \frac{\prod_{j \in \text{Des}(\pi)} z_0 z_{\pi(1)} z_{\pi(2)} \cdots z_{\pi(j)} \prod_{j \in \text{Neg}([\rho, \epsilon]\pi^{-1})} z_0 z_{\pi(1)} z_{\pi(2)} \cdots z_{\pi(j-1)}}{(1-z_0)(1-z_0 z_{\pi(1)} \cdots z_{\pi(n)}) \prod_{j=1}^{n-1} (1 - z_0^2 z_{\pi(1)}^2 \cdots z_{\pi(j)}^2)}.$$

This completes our proof, since from our triangulation it follows that

$$\sigma_{\text{cone}([0,1]^n)}(z_0, \dots, z_n) = \sum_{\pi \in S_n} \sigma_{\text{cone}(\Delta_\pi)}(z_0, \dots, z_n). \quad \square$$

*Proof of Theorem 6.2.* Setting  $t = z_0$  and  $q = z_1 = \cdots = z_n$  in Theorem 6.3 yields our desired form on the left-hand side of our identity, while the denominator of the right-hand side uniformly becomes

$$(1-t)(1-tq^n) \prod_{j=1}^{n-1} (1-t^2 q^{2j}).$$

Each element  $(\rho, \epsilon)\pi \in \bigcup_{\pi \in S_n} I_{2,n}\pi$  contributes to the numerator on the right-hand side of our identity a summand of

$$\prod_{j \in \text{Des}(\pi)} tq^j \prod_{j \in \text{Neg}([\rho, \epsilon]\pi^{-1})} tq^{j-1}.$$

Because  $\text{Des}(\pi) = \text{Des}_A((\rho, \epsilon)\pi)$ , it follows that

$$\prod_{j \in \text{Des}(\pi)} tq^j \prod_{j \in \text{Neg}([\rho, \epsilon]\pi^{-1})} tq^{j-1} = t^{\text{dndes}(\rho(\pi, \epsilon))} q^{\text{dnmaj}(\rho(\pi, \epsilon))},$$

hence our proof is complete.  $\square$

**6.3.  $q$ -analogues involving  $(2k+1)^n$ .** We derive in this subsection a type- $D$  analogue of Theorem 5.11, in the sense that we'd like to have the same left-hand side and the same denominator on the right-hand side. A natural starting place in this endeavor is to seek a  $q$ -analogue of Brenti's identity (7). We should mention that in [9], Brenti stipulated that he proved an " $s$ -analogue" of (7) (where  $s$  plays the same role as in Theorem 5.11), while Biagioli [6, Theorem 3.4] and Biagioli-Zeng [7, Theorem 6.3] computed different type- $D$  analogues.

In type  $D$ , the (possible) descent positions 0 and 1 are special; thus we define

$$\begin{aligned} N_2(\epsilon) &:= \# \{ \epsilon_j : j \geq 2, \epsilon_j = -1 \}, \\ \text{dnatdes}_2(\pi, \epsilon) &:= \# (\text{DNatDes}(\pi, \epsilon) \cap [2, n]), \\ \text{maj}_2(\pi, \epsilon) &:= \sum_{j \in \text{DNatDes}(\pi, \epsilon) \cap [2, n]} j. \end{aligned}$$

Recalling that  $B_n(k)$  denotes the  $n^{\text{th}}$  Bernoulli polynomial and referencing (7), since

$$B_n(k+1) - B_n(0) = n \sum_{j=0}^k j^{n-1},$$

we can simplify

$$\sum_{k \geq 0} 2^{n-1} (B_n(k+1) - B_n(0)) t^k = \frac{n 2^{n-1} t}{1-t} \sum_{j \geq 0} (j+1)^{n-1} t^j = \frac{n 2^{n-1} t \sum_{\pi \in S_{n-1}} t^{\text{des}(\pi)}}{(1-t)^{n+1}}.$$

Via the natural  $(n 2^{n-1})$ -to-one map  $D_n \rightarrow S_{n-1}$  given by  $(\pi, \epsilon) \mapsto (\epsilon_2 \pi(2), \epsilon_3 \pi(3), \dots, \epsilon_n \pi(n))$ , where we interpret  $(\epsilon_2 \pi(2), \epsilon_3 \pi(3), \dots, \epsilon_n \pi(n))$  as an element of  $S_{n-1}$ , we obtain

$$n 2^{n-1} \sum_{\pi \in S_{n-1}} t^{\text{des}(\pi)} = \sum_{(\pi, \epsilon) \in D_n} t^{\text{dnatdes}_2(\pi, \epsilon)}.$$

This gives the following reformulation of (7), which—as we will see—arises naturally from our approach:

$$(13) \quad \sum_{k \geq 0} (2k+1)^n t^k = \frac{\sum_{(\pi, \epsilon) \in D_n} t^{\text{dnatdes}(\pi, \epsilon)} + t^{1+\text{dnatdes}_2(\pi, \epsilon)}}{(1-t)^{n+1}}.$$

For a proposition  $P$ , we use the notation

$$[P] := \begin{cases} 1 & \text{if } P \text{ is true,} \\ 0 & \text{if } P \text{ is false.} \end{cases}$$

Our  $q$ -version of (13) is the following.

**Theorem 6.4.**

$$\begin{aligned} & \sum_{k \geq 0} ([k+1]_q + s[k]_q)^n t^k \\ &= \frac{\sum_{(\pi, \epsilon) \in D_n} s^{N_2(\epsilon)} t^{\text{dnatdes}_2(\pi, \epsilon)} q^{\text{maj}_2(\pi, \epsilon)} \left( (st)^{[\epsilon_1=-1]} (tq)^{[1 \in \text{DNatDes}(\pi, \epsilon)]} + (st)^{[\epsilon_1=1]} (tq)^{[0 \in \text{DNatDes}(\pi, \epsilon)]} \right)}{\prod_{j=0}^n (1-tq^j)} \\ &= \frac{\sum_{(\pi, \epsilon) \in D_n} s^{N_2(\epsilon)} t^{\text{dnatdes}_2(\pi, \epsilon)} q^{\text{maj}_2(\pi, \epsilon)} \left( (tq)^{[0 \text{ or } 1 \in \text{DNatDes}(\pi, \epsilon)]} + st(tq)^{[0 \text{ and } 1 \in \text{DNatDes}(\pi, \epsilon)]} \right)}{\prod_{j=0}^n (1-tq^j)}. \end{aligned}$$

Brenti's identity (13) follows upon setting  $s = q = 1$  and noticing that

$$t^{\text{dnatdes}_2(\pi, \epsilon)} \left( t^{[0 \text{ or } 1 \in \text{DNatDes}(\pi, \epsilon)]} + t \cdot t^{[0 \text{ and } 1 \in \text{DNatDes}(\pi, \epsilon)]} \right) = t^{\text{dnatdes}(\pi, \epsilon)} + t^{1+\text{dnatdes}_2(\pi, \epsilon)}.$$

As far as we can tell, no such simplification happens when we use a different specialization for  $s$  and  $q$ .

*Proof of Theorem 6.4.* For a given  $(\pi, \epsilon) \in D_n$ , let  $\tilde{\epsilon} := (-\epsilon_1, \epsilon_2, \dots, \epsilon_n)$ . Then

$$\bigcup_{(\pi, \epsilon) \in D_n} \{(\pi, \epsilon), (\pi, \tilde{\epsilon})\}$$

is the hyperoctahedral group, and so by Theorem 5.11,

$$\sum_{k \geq 0} ([k+1]_q + s[k]_q)^n t^k = \frac{\sum_{(\pi, \epsilon) \in D_n} (s^{N(\epsilon)} t^{\text{natdes}(\pi, \epsilon)} q^{\text{maj}(\pi, \epsilon)} + s^{N(\tilde{\epsilon})} t^{\text{natdes}(\pi, \tilde{\epsilon})} q^{\text{maj}(\pi, \tilde{\epsilon})})}{\prod_{j=0}^n (1-tq^j)}.$$

In terms of these descents,  $(\pi, \epsilon)$  and  $(\pi, \tilde{\epsilon})$  differ only (possibly) in positions 0 and 1, and so a summand in the numerator looks like

$$(14) \quad \begin{aligned} & s^{N(\epsilon)} t^{\text{natdes}(\pi, \epsilon)} q^{\text{maj}(\pi, \epsilon)} + s^{N(\tilde{\epsilon})} t^{\text{natdes}(\pi, \tilde{\epsilon})} q^{\text{maj}(\pi, \tilde{\epsilon})} \\ &= s^{N_2(\epsilon)} t^{\text{dnatdes}_2(\pi, \epsilon)} q^{\text{maj}_2(\pi, \epsilon)} \left( (st)^{[0 \in \text{DNatDes}(\pi, \epsilon)]} (tq)^{[1 \in \text{DNatDes}(\pi, \epsilon)]} + (st)^{[0 \in \text{NatDes}(\pi, \tilde{\epsilon})]} (tq)^{[1 \in \text{NatDes}(\pi, \tilde{\epsilon})]} \right). \end{aligned}$$

The role of  $s$  in the large parenthesis comes from the fact that  $\epsilon_1 = -1$  if and only if  $0 \in \text{NatDes}(\pi, \epsilon)$ . Furthermore, since

$$1 \in \text{NatDes}(\pi, \epsilon) \iff \epsilon_1 \pi_1 > \epsilon_2 \pi_2 \iff 1 \in \text{DNatDes}(\pi, \epsilon)$$

and

$$1 \in \text{NatDes}(\pi, \tilde{\epsilon}) \iff -\epsilon_1 \pi_1 > \epsilon_2 \pi_2 \iff 0 \in \text{DNatDes}(\pi, \epsilon),$$

we can rewrite (14) as

$$\begin{aligned} & s^{N(\epsilon)} t^{\text{natdes}(\pi, \epsilon)} q^{\text{maj}(\pi, \epsilon)} + s^{N(\tilde{\epsilon})} t^{\text{natdes}(\pi, \tilde{\epsilon})} q^{\text{maj}(\pi, \tilde{\epsilon})} \\ &= s^{N_2(\epsilon)} t^{\text{dnatdes}_2(\pi, \epsilon)} q^{\text{maj}_2(\pi, \epsilon)} \left( (st)^{[\epsilon_1 = -1]} (tq)^{[1 \in \text{DNatDes}(\pi, \epsilon)]} + (st)^{[\epsilon_1 = 1]} (tq)^{[0 \in \text{DNatDes}(\pi, \epsilon)]} \right). \square \end{aligned}$$

## 7. WREATH PRODUCTS

**7.1.  $q$ -analogues involving  $(k+1)^n$ .** We now consider statistics for the groups  $\mathbb{Z}_r \wr S_n$ ; these extend and subsume the statistics introduced for the type- $B$  case. We begin by defining type- $A$  statistics for elements of  $\mathbb{Z}_r \wr S_n$ , [3, 4]. Throughout, we use the total order from Definition 2.3 on the elements of  $\{\omega^{r-1}, \omega^{r-2}, \dots, \omega^0\} \times [n]$ , i.e.,  $j^{c_j} < k^{c_k}$  if  $c_j > c_k$  or if both  $c_j = c_k$  and  $j < k$  hold.

**Definition 7.1.** For an element  $(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n$ , we define the *negative set* of  $(\pi, \epsilon)$  to be

$$\text{Neg}(\pi, \epsilon) := \{i \in [n] : \epsilon_i \neq \omega^0 = 1\},$$

and we define  $\text{neg}(\pi, \epsilon) := \#\text{Neg}(\pi, \epsilon)$ . Writing  $\epsilon_j = \omega^{c_j}$ , we define the *color sum statistic* to be

$$\text{col}(\pi, \epsilon) := \sum_{i \in [n]} c_i.$$

The *type- $A$  descent set* is defined to be

$$\text{Des}_A(\pi, \epsilon) := \{i \in [n-1] : \pi_i^{c_i} > \pi_{i+1}^{c_{i+1}}\}$$

and the *type- $A$  descent statistic* is

$$\text{des}_A(\pi, \epsilon) := \#\text{Des}_A(\pi, \epsilon).$$

We next define *negative* statistics for wreath products, following [3].

**Definition 7.2.** For an element  $(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n$ , we define the *negative inverse multiset* as

$$\text{NNeg}(\pi, \epsilon) := \{i^{c_i} : i \in [n]\}.$$

We define the *negative descent multiset* as

$$\text{NDes}(\pi, \epsilon) := \text{Des}_A(\pi, \epsilon) \cup \text{NNeg}((\pi, \epsilon)^{-1}).$$

The *negative descent statistic* is

$$\text{ndes}(\pi, \epsilon) := \#\text{NDes}(\pi, \epsilon).$$

The *negative major index* is

$$\text{nma}(\pi, \epsilon) := \sum_{i \in \text{NDes}(\pi, \epsilon)} i.$$

Observe that  $\text{NNeg}((\pi, \epsilon)^{-1})$  contains exactly  $(r - c_{\pi^{-1}(i)}) \bmod r$  copies of each  $i \in [n]$ . The negative statistics for wreath products lead to the following theorem due to Bagno [3].

**Theorem 7.3** (Bagno).

$$\sum_{k \geq 0} [k+1]_q^n t^k = \frac{\sum_{(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n} t^{\text{ndes}(\pi, \epsilon)} q^{\text{nma}(\pi, \epsilon)}}{(1-t) \prod_{j=1}^n (1-t^r q^{rj})}.$$

There are also flag statistics for wreath products, due to Bagno and Biagioli [4].

**Definition 7.4** (Bagno–Biagioli). For an element  $(\pi, \epsilon)$  in  $\mathbb{Z}_r \wr S_n$ , we define the *flag descent statistic* as

$$\text{fdes}(\pi, \epsilon) := r \cdot \text{des}_A(\pi, \epsilon) + c_1,$$

where as usual  $\epsilon_1 = \omega^{c_1}$ . The *flag major index* is

$$\text{fma}(\pi, \epsilon) := r \cdot \text{major}_A(\pi, \epsilon) + \text{col}(\pi, \epsilon).$$

The flag statistics for wreath products lead to the following theorem.

**Theorem 7.5** (Bagno–Biagioli).

$$\sum_{k \geq 0} [k+1]_q^n t^k = \frac{\sum_{(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n} t^{\text{fdes}(\pi, \epsilon)} q^{\text{fma}(\pi, \epsilon)}}{(1-t) \prod_{j=1}^n (1-t^r q^{rj})}.$$

Bagno and Biagioli prove in [4] a multivariate theorem of this type for a family of normal subgroups of  $\mathbb{Z}_r \wr S_n$ . Their techniques involve studying colored-descent representations of these subgroups, which are representations of the groups on the associated coinvariant algebra.

**7.2. Multivariate identities.** We state in this subsection new multivariate generalizations of Theorems 7.3 and 7.5. When our theorems arise from straightforward modifications of our proofs in type  $B$ , we will sketch or omit their proofs.

**Definition 7.6.** Define the subset of *increasing elements* of  $\mathbb{Z}_r \wr S_n$ , denoted  $I_{r,n}$ , to be those elements satisfying  $\text{des}_A(\rho, \epsilon) = 0$ , i.e.,  $I_{r,n}$  contains all permutations  $(\rho, \epsilon)$  such that  $\rho(j)^{\epsilon_j} < \rho(j+1)^{\epsilon_{j+1}}$ .

It is straightforward that every element of  $\mathbb{Z}_r \wr S_n$  can be represented uniquely as  $(\rho, \epsilon)\pi = (\rho\pi, \epsilon_\pi)$  for some  $\pi \in S_n$  and  $(\rho, \epsilon) \in I_{r,n}$ , where  $\rho\pi$  is the standard composition of permutations and  $\epsilon_\pi = (\epsilon_{\pi(1)}, \epsilon_{\pi(2)}, \dots, \epsilon_{\pi(n)})$ . Thus,

$$\mathbb{Z}_r \wr S_n = \bigcup_{\pi \in S_n} I_{r,n}\pi.$$

Observe that  $\text{Neg}([( \rho, \epsilon)\pi]^{-1}) = \text{Neg}((\rho, \epsilon)^{-1})$ . Our multivariate version of Theorem 7.3 is the following.

**Theorem 7.7.**

$$\sum_{k \geq 0} \prod_{j=1}^n (1 + z_j + z_j^2 + \cdots + z_j^k) z_0^k = \sum_{\pi \in S_n} \sum_{(\rho, \epsilon) \in I_{r,n}} \frac{\prod_{j \in \text{Des}(\pi)} z_0 z_{\pi(1)} z_{\pi(2)} \cdots z_{\pi(j)} \prod_{j \in \text{Neg}((\rho, \epsilon)^{-1})} z_0 z_{\pi(1)} z_{\pi(2)} \cdots z_{\pi(j)}}{(1 - z_0) \prod_{j=1}^n (1 - z_0^r z_{\pi(1)}^r \cdots z_{\pi(j)}^r)}.$$

The following definition is needed to state our multivariate version of Theorem 7.5.

**Definition 7.8.** For an element  $\epsilon = (\omega^{c_j})_{j=1}^n \in \{1, \omega^1, \omega^2, \dots, \omega^{r-1}\}^n$  with  $\epsilon_{n+1} := 1 = \omega^0$ , for  $j \in [n]$  define

$$a_j^\epsilon := (c_j - c_{j+1}) \pmod{r},$$

which we call the  $j$ th *sign change* for  $\epsilon$ . Define  $\text{ch}(s) := \sum_j a_j^\epsilon$  to be the *total sign change* in  $\epsilon$ .

**Theorem 7.9.**

$$\sum_{k \geq 0} \prod_{j=1}^n (1 + z_j + z_j^2 + \cdots + z_j^k) z_0^k = \sum_{(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n} \frac{\prod_{\substack{j \in \text{Des}(\pi) \\ a_j^\epsilon = 0}} z_0^r z_{\pi(1)}^r z_{\pi(2)}^r \cdots z_{\pi(j)}^r \prod_{j=1}^n z_0^{a_j^\epsilon} z_{\pi(1)}^{a_j^\epsilon} z_{\pi(2)}^{a_j^\epsilon} \cdots z_{\pi(j)}^{a_j^\epsilon}}{(1 - z_0) \prod_{j=1}^n (1 - z_0^r z_{\pi(1)}^r z_{\pi(2)}^r \cdots z_{\pi(j)}^r)}.$$

The proofs of Theorems 7.7 and 7.9 together yield a bijective proof of the equidistribution of the pairs of statistics  $(\text{ndes}, \text{nmaj})$  and  $(\text{fdes}, \text{fmajor})$  for  $\mathbb{Z}_r \wr S_n$ . As far as we know, this bijection is new. The proof of this bijection follows immediately from the proofs of Theorems 7.7 and 7.9, whose proofs are easily obtained by modifying the proofs of Theorems 5.7 and 5.9.

**Corollary 7.10.**

$$\sum_{(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n} t^{\text{ndes}(\pi, \epsilon)} q^{\text{nmaj}(\pi, \epsilon)} = \sum_{(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n} t^{\text{fdes}(\pi, \epsilon)} q^{\text{fmajor}(\pi, \epsilon)}.$$

**7.3.  $q$ -analogues involving arbitrary  $(rk+1)^n$ .** In [12, Theorem 9], Chow and Mansour provide an Euler–Mahonian distribution for wreath products using Steingrímsson’s descent statistics and a new flag major index. We state in this section a similar Euler–Mahonian distribution for the descent statistic and flag major index given in Definitions 2.3 and 7.4. By combining Theorem 7.11 below and [12, Theorem 9], we see that the pairs

(Steingrímsson’s descent statistic, Chow–Mansour’s flag major index)

and

(des, fmajor)

are equidistributed over  $\mathbb{Z}_r \wr S_n$ .

**Theorem 7.11.**

$$\sum_{k \geq 0} [rk+1]_q^n t^k = \frac{\sum_{(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n} t^{\text{des}(\pi, \epsilon)} q^{\text{fmajor}(\pi, \epsilon)}}{\prod_{j=0}^n (1 - tq^{rj})}.$$

We obtain in Theorem 7.13 below a multivariate generalization of our bivariate identity. At the end of [12], Chow and Mansour indicate that having a Hilbert-series interpretation of [12, Theorem 9] is an open problem. The proof of Theorem 7.13 provides such an interpretation, after taking into account Remark 4.3.

*Remark 7.12.* In [8, Equation (8.1)], Biagioli and Zeng obtain a generalization of the Chow–Gessel Theorem 5.11 for wreath products. The authors do not at this time know of a way to obtain this identity using polyhedral geometry.

#### 7.4. Multivariate identities.

**Theorem 7.13.**

$$\sum_{k \geq 0} \prod_{j=1}^n \left( 1 + z_j + z_j^2 + \cdots + z_j^{2k} \right) z_0^k = \sum_{(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n} \frac{z_0^{\lceil \text{ch}(\epsilon)/r \rceil} \prod_{j=1}^n z_{\pi(1)}^{a_j^\epsilon} z_{\pi(2)}^{a_j^\epsilon} \cdots z_{\pi(j)}^{a_j^\epsilon} \prod_{\substack{j \in \text{Des}(\pi) \\ a_j^\epsilon = 0}} z_0 z_{\pi(1)}^r z_{\pi(2)}^r \cdots z_{\pi(j)}^r}{\prod_{j=0}^n \left( 1 - z_0 z_{\pi(1)}^r z_{\pi(2)}^r \cdots z_{\pi(j)}^r \right)}.$$

*Proof.* As in our previous proofs, this proof proceeds in two stages. We first triangulate the cube  $[0, r]^n$  into a disjoint union of simplices, then set up an indexing system for the integer points in the fundamental parallelepipeds for the cones over these simplices. Second, we bijectively associate the elements of  $\mathbb{Z}_r \wr S_n$  with these integer points in a way that allows us to recover, in our subsequent proof of Theorem 7.11, the descent and flag major statistics from these integer points.

We begin by triangulating  $[0, r]^n$  into the disjoint simplices

$$\Delta_\pi := \left\{ \mathbf{x} \in \mathbb{R}^n : \begin{array}{l} 0 \leq x_{\pi(n)} \leq x_{\pi(n-1)} \leq \cdots \leq x_{\pi(1)} \leq r, \\ x_{\pi(j+1)} < x_{\pi(j)} \text{ if } j \in \text{Des}(\pi) \end{array} \right\}$$

(one for each  $\pi \in S_n$ ). As before, the strict inequalities determined by the descent set of  $\pi$  ensures that this triangulation is disjoint.

Unlike the cones produced by coning over the simplices in our triangulation of  $[0, 1]^n$ , the cones arising from this triangulation of  $[0, r]^n$  are not unimodular. By Lemma 3.2, the integer-point transform of  $\text{cone}(\Delta_\pi)$  can be expressed as a rational function where the denominator has the form

$$\prod_{j=0}^n \left( 1 - z_0 z_{\pi(1)}^r z_{\pi(2)}^r \cdots z_{\pi(j)}^r \right),$$

i.e., where the displayed exponent vectors are the ray generators for this cone. As the determinant of the matrix formed by the ray generators of  $\text{cone}(\Delta_\pi)$  is  $r^n$ , there are  $r^n$  integer points in the fundamental parallelepiped of  $\text{cone}(\Delta_\pi)$ . It is a straightforward observation that there are  $r^n$  such integer points formed by taking linear combinations of the ray generators for the cone with coefficients from the set  $\{0, \frac{1}{r}, \dots, \frac{r-1}{r}\}$ . We will use the following notation to denote such an integer point; for  $\alpha_j \in \{0, \frac{1}{r}, \dots, \frac{r-1}{r}\}$  where  $j = 0, \dots, n$ ,

$$\mathbf{p} = \alpha_0 \mathbf{e}_0 + \sum_{j=1}^n \alpha_j \left( \mathbf{e}_0 + r \mathbf{e}_{\pi(1)} + r \mathbf{e}_{\pi(2)} + \cdots + r \mathbf{e}_{\pi(j)} \right).$$

To determine the numerator of  $\sigma_{\text{cone}(\Delta_\pi)}(z_0, \dots, z_n)$ , we must shift some of these interior points off of the boundary of  $\text{cone}(\Delta_\pi)$ . It is a simple exercise to show that the point  $\mathbf{p}$  must be shifted by

the vector

$$\sum_{\substack{j \in \text{Des}(\pi) \\ \alpha_j = 0}} \mathbf{e}_0 + r\mathbf{e}_{\pi(1)} + r\mathbf{e}_{\pi(2)} + \cdots + r\mathbf{e}_{\pi(j)}.$$

Thus,

$$\sigma_{\text{cone}(\Delta_\pi)}(z_0, z_1, \dots, z_n) = \frac{z_0^{\sum_{j=0}^n \alpha_j} \prod_{j=1}^n z_{\pi(1)}^{r\alpha_j} z_{\pi(2)}^{r\alpha_j} \cdots z_{\pi(j)}^{r\alpha_j} \prod_{\substack{j \in \text{Des}(\pi) \\ \alpha_j = 0}} z_0 z_{\pi(1)}^r z_{\pi(2)}^r \cdots z_{\pi(j)}^r}{\prod_{j=0}^n \left(1 - z_0 z_{\pi(1)}^r z_{\pi(2)}^r \cdots z_{\pi(j)}^r\right)}.$$

Note that  $\sum_{j=0}^n \alpha_j$  is always an integer, as this is the coefficient of  $\mathbf{e}_0$  in the resulting integer point.

We now associate to the element  $(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n$  the integer point

$$\alpha_0 \mathbf{e}_0 + \sum_{j=1}^n \frac{a_j^\epsilon}{r} (\mathbf{e}_0 + r\mathbf{e}_{\pi(1)} + r\mathbf{e}_{\pi(2)} + \cdots + r\mathbf{e}_{\pi(j)}) + \sum_{\substack{j \in \text{Des}(\pi) \\ a_j^\epsilon = 0}} (\mathbf{e}_0 + r\mathbf{e}_{\pi(1)} + r\mathbf{e}_{\pi(2)} + \cdots + r\mathbf{e}_{\pi(j)})$$

in  $\text{cone}(\Delta_\pi)$ , where  $\alpha_0$  takes the value needed to make  $\alpha_0 + \sum_{j=1}^n \frac{a_j^\epsilon}{r}$  an integer. Through this

association, the set of sign vectors  $\{1, \omega, \omega^2, \dots, \omega^{r-1}\}^n$  parametrizes the integer points in the fundamental parallelepiped for  $\text{cone}(\Delta_\pi)$ . This parametrization is bijective, and the coefficient of  $\mathbf{e}_0$  in the first two terms of the sum above is equal to both  $\lceil \text{ch}(\epsilon)/r \rceil$  and  $\sum_{j=0}^n \alpha_j$ . Thus, our proof is complete following the observation that

$$\sigma_{\text{cone}([0, r]^n)}(z_0, \dots, z_n) = \sum_{\pi \in S_n} \sigma_{\text{cone}(\Delta_\pi)}(z_0, \dots, z_n). \quad \square$$

Rather than include a detailed proof of how Theorem 7.11 follows from Theorem 7.13, we will instead sketch the proof and then include an illustrative example.

*Proof of Theorem 7.11.* Setting  $t = z_0$  and  $q = z_1 = \cdots = z_n$  in Theorem 7.13 yields our desired form on the left-hand side, while the denominator of the right-hand side uniformly becomes  $\prod_{j=0}^n (1 - tq^{rj})$ . Each element  $(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n$  contributes a summand of

$$t^{\lceil \text{ch}(\epsilon)/r \rceil} \prod_{j: a_j^\epsilon \neq 0} q^{a_j^\epsilon j} \prod_{\substack{j \in \text{Des}(\pi) \\ a_j^\epsilon = 0}} tq^{rj}$$

to the numerator of the right-hand side. Hence, we need to prove

$$(15) \quad \text{des}(\pi, \epsilon) = \lceil \text{ch}(\epsilon)/r \rceil + \#\{j \in \text{Des}(\pi) : a_j^\epsilon = 0\},$$

and

$$(16) \quad \text{fmajor}(\pi, \epsilon) = \sum_{j: a_j^\epsilon \neq 0} a_j^\epsilon j + \sum_{\substack{j \in \text{Des}(\pi) \\ a_j^\epsilon = 0}} rj.$$

As explained in the proof of Theorem 5.5 in the case of type  $B$ , a descent in position  $j$  of  $(\pi, \epsilon)$  can arise for one of three reasons: either  $c_j > c_{j+1}$ , or  $c_j = c_{j+1}$  and  $j \in \text{Des}(\pi)$ , or  $j = 1$  and  $c_1 \neq 0$ .



When viewing  $(\pi, \epsilon)$  in window notation and reading right to left, every descent of the first kind is the result of successive sign changes that sum to a value greater than or equal to  $r$ . Descents of the second kind are caused when there is no sign change but a descent in  $\pi$  is present. Descents of the third kind are caused by an “incomplete” sign change in  $\epsilon$ , i.e., where non-zero colors are introduced when reading window notation right to left, but these colors are not returned to a 0 before reaching the first position on the left. The equality in (15) follows immediately from these observations.

To prove (16), observe that  $\sum_{\substack{j \in \text{Des}(\pi) \\ a_j^\epsilon = 0}} rj$  accounts in the type- $A$  major index for the contribution

given by those descents that do not involve a sign change; it then only remains to consider the contribution of those descents arising from a sign change. Therefore, what remains to be shown is that

$$(17) \quad \sum_{j: a_j^\epsilon \neq 0} a_j^\epsilon j = \text{col}(\pi, \epsilon) + \sum_{\substack{j \in \text{Des}(\pi, \epsilon) \\ \epsilon_j \neq \epsilon_{j+1}}} rj.$$

We will sketch a proof of this and include an example illustrating it.

In order for a descent to arise as a sign change, it must have been that reading right to left in window notation for  $(\pi, \epsilon)$ , the sum of the sign changes to that point must have exceeded  $r - 1$ , hence returning the color for the descent position to a value less than the one preceding it on the right. Therefore, if the descent occurred in position  $d$ , the term

$$\sum_{j: a_j^\epsilon \neq 0} a_j^\epsilon j$$

will have contributed  $rd$  for that descent plus some additional amount. This additional amount then becomes distributed between the term  $\text{col}(\pi, \epsilon)$  for the colors  $c_i$  where  $i \geq d$ , and the sum  $rd'$  for the next descent position  $d'$  to the left of  $d$ . Hence, (17) holds, and our proof is complete.  $\square$

*Example 7.14.* To illustrate the final step in the proof above, we will show that for the color vector  $c = 412301131$  arising in a possible element of  $\mathbb{Z}_5 \wr S_9$ , (17) holds. Note that there are four type- $A$  descents in  $c$  caused by sign changes, occurring in positions 7, 5, 3, and 2. The sign change vector for  $c$  is  $a = 344340321$ , where the right-hand 1 accounts for  $c_9 - c_{10}$  where  $c_{10}$  is by definition 0. We then have that

$$9a_9 + 8a_8 + 7a_7 = 7 \cdot 5 + c_9 + c_8 + 7.$$

Hence, the descent in position 7 contributes  $7 \cdot 5 + c_9 + c_8$  to the left-hand side of (17), plus 7 more which is counted in

$$7 + 5a_5 = 5 \cdot 5 + c_7 + c_6.$$

This yields the contribution of the 5th descent position plus colors  $c_6$  and  $c_7$  to the left-hand side of (17). Next we obtain that

$$4a_4 + 3a_3 = 3 \cdot 5 + c_5 + c_4 + 6,$$

contributing the 3rd descent position plus colors  $c_5$  and  $c_4$  to (17) with 6 remaining. This leftover 6 then contributes to

$$6 + 2a_2 = 2 \cdot 5 + c_3 + 2,$$

where the final 2 in the expression above contributes to

$$2 + 1a_1 = c_2 + c_1.$$

Thus, the sum of  $ja_j$  over  $j$  is equal to the statistic  $\text{col}(\pi, \epsilon)$  plus the terms  $rj$  in the type- $A$  major index statistic arising from sign changes rather than descents in  $\pi$ .

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